Pricing local search engines for company websites

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Abstract

Ever since the dramatic boom of the Internet, web search engines have appealed to both researchers’ and developers’ attention to a noticeable pitch. In order to enhance the website efficiency, a number of companies rent the company-wide local search engines from the search engine providers. Since web service renting serves as a promising alternative of web service purchase within the web industrial framework, how to effectively price such kind of search engines becomes an important as well as impending issue, with which practitioners and researchers confront. In this paper, we present a pricing model, which is based on the discrete-time independent incremental process, for the local search engines of the company website. The stopping time is defined in this work and the expected revenue for the web-search-engine providers over the rental horizon is also derived. Due to the considerable complexity and difficulty to obtaining an analytical solution for estimation of the expected revenue, the optimal monthly rental is discussed and exemplified through empirical experiments. By maximizing the revenue, two different strategies are investigated by allowing different initial lock-in periods and offering a coupon for waiving certain amount of fee for initial use. The experiments illustrate the best rental and sale price scenarios.

Keywords: Search engine pricing; Web site; Stopping time; Independent incremental process; Initial lock-in periods; Coupons

1. Introduction

Under the swift development of the Internet and eCom-merce, there has been an increasingly growing demand for web search engines during the past several years. As a matter of fact, web pages have gradually replaced those traditional media means thus becoming the most favorable information resource for a lot of people. It’s worth noticing that some high-profile software providers, e.g., the Microsoft, have also started crossing over into and even segmenting the market of web search engines.

Currently, many companies provide search features for their customers to browse and search at their websites. Since the search engine module is not tightly coupled to the building of the website, many companies either lease or purchase the search engines and simply incorporate them with their websites, instead of understanding the codes themselves. Most importantly, the search engines help customers to solve their problems so only fewer customers would need to call the customer service department for help, and that reduces the cost of customer service for the company. In addition, web search engines also facilitate the customers by curtailing their time spent on browsing the huge website, which may alleviate the web server burden.

In this paper, we propose the best scenario that the search-engine providers should offer. On one hand, a search-engine providers are eager to gain a higher profit, hence increasing the price to achieve this goal; on the other hand, they want to keep a certain amount of lessees, and thus reducing the price seems to be a better choice. In order to solve such conflict, we need to strike a balance in between and try to optimize the price for search-engine providers.

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The search engine is a sort of software that was born with the web and applied to the web. Pricing search engines is involved in many factors and is a novel topic in the eCommerce economics. There have been several approaches for software pricing, the monopoly-based approach, the value-based approach, the utility-based approach, and the supply-demand-based approach. In practice, the software providers generally use a combination of the aforementioned approaches. For example, considering the existence of a surplus of the web service supply, the queue theory was used in Cheng and Koehler [1] to price the software. Also, for the similar software, the providers with reputation could offer lower prices than the average. In some cases, it is acknowledged that renting software is a good option. Basically, the rental could be determined based on the number of customers, or the usage, or a combination of the two. Sometimes, the companies would to first rent the software while having an option to purchase it later. Wilcox and Farmer [2] considered the issue from the consumer side and discovered that it is less expensive to rent the software for the first two years and then purchase it. Gurnani and Karlapalnen [6] employed the linear programming in pricing software and discovered that the providing of monthly rental choices could increase the market size, hence enhance the revenue for the software providers. Furthermore, Cheng and Koehler [1], Varian [3], Choudhary and Tomak [4] and Bhargava et al. [5] gave investigations. It is worth mentioning that currently a proportion of the web search engines are free, nonetheless due to the potential profits search engines can bring to the companies in the long run, the business model explored in this paper is expected to be of considerable value when the market for search engines is ready.

The remainder of this paper is organized as follows. In Section 2, the conditions for a lessee to start renting and to keep the rental are derived. Section 3 optimizes the rental from the provider’s side. Considering the complexity of the pricing model, the strategies of experimental study are presented in Section 4. Section 5 further implements the strategies in Section 4. Section 6 concludes the paper.

2. Condition for a lessee to keep rental

In this section, under different conditions, from the perspective of the providers, a lessee who keeps the rental is investigated. Our target is to explore the favorable conditions regarding purchasers to maximize the revenue for the providers.

Let \( C_0 \) be the average cost before using the search engine, \( C_k \) the cost in the \( k \)-th month after using the search engine, \( k = 1, 2, 3, \ldots \), then \( X_k = C_0 - C_k \) is the saving in the \( k \)-th month.

We assume that \( C_0 \) is a normal random variable with the mean \( \mu_0 \) and the variation \( \sigma_0^2 \), namely \( C_0 \sim (N(\mu_0, \sigma_0^2)) \) and \( C_k \) is an independent identically distributed (i.i.d.) normal random variable with the mean \( \mu_k \) and the variation \( \sigma_k^2 \), thus \( C_k \sim N(\mu_k, \sigma_k^2) \), then it can be achieved that

\[
X_k = C_0 - C_k, \quad k = 1, 2, 3, \ldots, \text{ is the normal random variable with the mean } \mu = \mu_0 - \mu_k > 0, \text{ and the variation } \sigma^2 = \sigma_0^2 + \sigma_k^2 > 0, \text{ i.e. } X_k \sim (N(\mu, \sigma^2)), \quad k = 1, 2, 3, \ldots
\]

Furthermore, suppose that \( w \) is the monthly rental, paid on the 1st day of each month, \( D \) is the initial lock-in period, i.e., the least months of rental, if the lessee is to rent, \( R \) is the reservation value of the lessee, and \( H \) is the interested period of the search engine (after a period of time, the software is obsolete).

Next, we derive the condition for the lessee to start renting.

Clearly, the monthly rental should be less than the amount of saving each month, thus \( w < X_k, \quad k = 1, 2, 3, \ldots \), or \( 0 < X_k - w \). However, this is not attractive enough to encourage the lessee to start renting. On the other hand, if at the end of the \( k \)-th month, the saving amount \( X_k - w \) is larger than the reservation value of the lessee, namely

\[
R < X_k - w, \quad (1)
\]

then the lessee is willing to start renting. We define

\[
R = \theta X_k, \quad (2)
\]

where \( \theta \) is the coefficient of the reservation value, in general, \( \theta \in (0, 1) \). Hence, for \( k = 1, 2, 3, \ldots, \)

\[
\theta X_k < X_k - w, \quad (3)
\]

\[
w < (1 - \theta) X_k, \quad (4)
\]

\[
(w <) \frac{w}{1 - \theta} < X_k. \quad (5)
\]

Apparently, if the above equations are satisfied, the lessee starts to lease the search engine. Taking the expectation on both sides we obtain

\[
E\left\{ \frac{w}{1 - \theta} \right\} < E X_k, \quad k = 1, 2, 3, \ldots, \quad (6)
\]

or

\[
\frac{w}{1 - \theta} < \mu. \quad (7)
\]

To attract the lessees, the expected monthly saving should be larger than \( \frac{w}{1 - \theta} \). Assume that the lessee starts to lease the search engine in month 0. Next we infer the conditions under which the lessee keeps leasing the search engine.

Let \( S_k = \sum_{k=0}^{K} X_k \) be the total saving from month 1 to month \( K \). Therefore, based on the previous assumption, \( S_k \) exhibits the normal distribution \( N(\mu K, \sigma^2 K) \), and \( \{S_k\} \) is an independent incremental Gaussian process.

We further assume that after the lessee starts to lease the search engine, as long as the average monthly saving \( \frac{S_k}{K} \) from month 1 to month \( K \) satisfies

\[
\frac{w}{1 - \theta} \leq \frac{S_k}{K}, \quad D \leq K \leq H, \quad (8)
\]

the lessee maintains the lease. Equivalently, it can be rewritten as

\[
\frac{w}{1 - \theta} K \leq S_k. \quad (9)
\]
If $\frac{w}{1-\theta}K \geq S_K$, the lessee stops using the search engine. Taking the expectation on both sides, we have

$$E\left\{\frac{w}{1-\theta}K\right\} < ES_K,$$

(10)

or

$$\frac{w}{1-\theta}K < \mu K.$$  

(11)

It announces that if a lessee is to be attracted, the total saving should be larger than $\frac{w}{1-\theta}K (> wK)$.

The sample paths of the stochastic process $S_K$ are exemplified in Fig. 1. In the initial lock-in period, even if $S_K$ is below the slope, the lessee must keep the lease. After the initial lock-in period, to keep the lessee, $\frac{w}{1-\theta}K < S_K$ must hold with $\frac{w}{1-\theta}K$ as the slope. If $S_K$ is always above the slope, the lessee shall keep the lease. If $S_K \leq \frac{w}{1-\theta}K$, or below the slope, the lessee stops leasing the search engine.

It can be evidently discovered that in the first month, the lessor is more inclined to lose the lessees. This is the primary reason why an initial lock-in period is designed. In the beginning, the trajectory of the stochastic process is very close to the slope. Nevertheless, due to the fact that it has a positive drift (0 < w < $\mu$), the stochastic process leaves the slope further as the time goes on and the chance of hitting the slope declines.

Offering a coupon for waiving certain amount of fee in the first month serves as another pricing strategy to avoid the loss of lessees in the beginning. Because in the first month, the coupon deducts the rental cost for the lessee therefore the probability for the lessee to stop renting the search engine is decreased at the beginning stage.

3. Model for best rental

For the provider, there are two pricing strategies that should be taken into account for leasing the local search engine to attract more lessees, which are (1) setting an initial lock-in period of length $D$, and

(2) offering a coupon for waiving certain amount of fee in the first month.

In the first strategy, the lessee must rent the local search engine for a prescribed period. For example, the initial lock-in period could be one quarter, or half of a year.

In the second strategy, the local search engine provider offers the lessee a coupon for waiving certain amount of fee. The provider could offer, for instance, a coupon for waiving the rental of one month. Namely, this scheme is to allow the lessee to use the local search engine free in the first month.

3.1. Best rental for strategy of initial lock-in period

Considering the pricing strategy with the initial lock-in period, we define the stopping time (the time that the lessee stops renting the local search engine) [7–10] of the stochastic process with regard to the area under the slope as

$$T(w) = \inf\{D \leq K \leq H : S_K \leq awK\},$$

(12)

where $a = \frac{1}{1-\theta}$. If $S_K > awK$, $D \leq K \leq H$, namely, the stochastic process of the search engine is never under the slope, let $T(w) = H$. As a result, $T(w)$ is either the first time that the stochastic process $S_K$ hits the slope $K$, or $H$. $T(w)$ is therefore the time that the lessee stops leasing the search engine.

We make a little change to the definition of $S_K$ and $T(w)$ so that the subindex of them starts from 1 for convenience. Let $X_1$ denote the saving of company in the initial lock-in $D$ months, and $X_k$, $k = 2, 3, 4, \ldots$, the saving in the $(D+k-1)$th month. Let $S_K = \sum_{k=1}^{D+k-1}X_k$, and $T(w)$ is defined as

$$T(w) = \inf\{1 \leq K \leq H - D + 1 : S_K \leq a(D + K - 1)w\}.$$  

(13)
The expected rental revenue of the search engine provider is

\[ Q(w) = \sum_{k=1}^{D} d^k w + E \left[ \sum_{k=2}^{T(w)-1} d^{D+k-1} w \right], \tag{14} \]

where \( d \) is the monthly discount factor.

Next, we derive the above model. From

\[ Q(w) = \sum_{k=1}^{D} d^k w + E \left[ \sum_{k=2}^{T(w)-1} d^{D+k-1} w \right] \]

\[ = \sum_{k=1}^{D} d^k w + w \left[ \sum_{k=2}^{H-D+1} d^{D+k-1} P\{k < T(w)\} \right] + 0P\{k \geq T(w)\} \]

\[ = \sum_{k=1}^{D} d^k w + w \left[ \sum_{k=2}^{H-D+1} d^{D+k-1} P\{k < T(w)\} \right]. \tag{15} \]

where the indicator function is

\[ I\{s < t\} = \begin{cases} 1, & s < t, \\ 0, & s \geq t. \end{cases} \tag{16} \]

Hence

\[ Q(w) = \sum_{k=1}^{D} d^k w \]

\[ + w \left[ \sum_{k=2}^{H-D+1} d^{D+k-1} [1P\{k < T(w)\} + 0P\{k \geq T(w)\}] \right] \]

\[ = \sum_{k=1}^{D} d^k w + w \left[ \sum_{k=2}^{H-D+1} d^{D+k-1} P\{k < T(w)\} \right]. \tag{17} \]

To calculate the best monthly rental, in the pricing strategy with the initial lock-in period, the principal or the lessor should optimize its utility or the following equation with regard to \( w \),

\[ w^* = \arg \max_w Q_k(w) \]

\[ = \arg \max_w \left\{ \sum_{k=1}^{D} d^k w + w \left[ \sum_{k=2}^{H-D+1} d^{D+k-1} P\{k < T(w)\} \right] \right\}. \tag{18} \]

Solving the above equation leads to the best monthly rental \( w^* \).

It can be seen from the above if the lessee changes the idea and wants to buy the search engine at time \( k \), the price he or she will be given is exactly the summation of the present values at time \( k \) of remaining rentals before the stopping time. In other words, whether she rents or buys does not change the amount of money he or she spends.

3.2. Best rental for strategy of offering a coupon of waiving certain amount of fee in the first month

The pricing strategy is to provide a coupon for waiving certain amount of fee for initial usage in the first month, say, decreasing the initial rental to \( w - v \), and keeping the monthly rental as \( w \) afterwards. The stopping time \( T(w,v) \) can be represented as

\[ T(w,v) = \inf \{ 1 \leq K \leq H : S_K \leq awK - av \}, \tag{19} \]

where \( v \) is the amount of waiver of initial fee, and \( awK - av \) is the total cost of customer in the first \( K \) months.

The expected rental revenue of the search engine provider is therefore

\[ Q'(w,v) = E \left[ \sum_{k=1}^{T(w,v)-1} d^k w \right] - v \]

\[ = wE \left[ \sum_{k=1}^{H-D+1} d^{D+k-1} E[I\{k < T(w,v)\}] \right] - v \]

\[ = wE \left[ \sum_{k=1}^{H-D+1} d^{D+k-1} P\{k < T'(w,v)\} \right] - v. \tag{20} \]

Likewise, in this strategy, the lessor should optimize the following equation with regard to \( w \) and \( v \),

\[ (w^*,v^*) = \arg \max_{w,v} Q'(w,v) \]

\[ = \arg \max_{w,v} \left\{ wE \left[ \sum_{k=1}^{H-D+1} d^{D+k-1} P\{k < T'(w,v)\} \right] - v \right\}. \tag{21} \]

Solving the above equation leads to the best monthly rental \( w^* \) and the best coupon value \( v^* \) in the first month.

If we set purchasing price as \( Q(w) \) in the first strategy and as \( Q'(w,v) \) in the second strategy, according to the models, no matter the lessee rents or buys, the revenues for the lessor remain the same. Consequently, there is no arbitrage opportunity.

It can also be seen from the above that if the lessee changes his or her idea and wants to buy the search engine at time \( k \), the price he or she will be given is also exactly the summation of the present values at time \( k \) of remaining rentals before the stopping time.

4. Algorithms for finding the best rental numerically

Due to the fact that the only difference between the aforementioned two methods lies in the two different definitions for the stopping time, without loss of generality, we apply the same notation \( G \) as the stopping time. Now we discuss the methods to get the best monthly rental \( w^* \) and the best coupon value \( v^* \).

We can find that the two probabilities about hitting time in \( Q(w) \) and \( Q'(w,v) \) can be expressed by some independent incremental Gauss process. Let \( G \) be stopping time of an independent incremental Gauss process, \( \{S_k\} \). Let \( W_k \) be the \( k \)th upper bound of in the definition of the stopping time \( G \), we have

\[ \{G > k\} = \{S_1 > W_1, S_2 > W_2, \ldots, S_k > W_k\}. \tag{22} \]

Clearly, \( P\{G > k\} \) is a \( k \)-dimensional integral. It is difficult to calculate the probability directly when \( k \) is large. It is also where the crux lies to work out the expected revenue.
and the best rental. Now we turn to the topic of numerical methods of calculating the best rentals for the two pricing strategies.

**Lemma 1.** Let \( X_1 \sim N(\mu_1, \sigma_1^2) \), \( X_k \sim N(\mu_k, \sigma_k^2) \), \( k = 2, 3, \ldots \), be independent identically distributed variables, and \( S_K = \sum_{k=1}^{K} X_k, M_K = \sum_{k=1}^{K} \mu_k \), \( V_K = \sum_{k=1}^{K} \sigma_k^2 \). For any \((S_{k-1}, S_k)\), \( k > 1\), we have the following results.

(a) \( S_k \sim N(M_k, V_k) \).

(b) For any \( k > 1 \), given \( S_{k-1} = s_{k-1} \), the conditional probability density of \( S_k \) is

\[
\frac{1}{\sqrt{2\pi\sigma_k^2}} \exp \left[ -\frac{(s_k - s_{k-1})^2}{2\sigma_k^2} \right].
\]

(c) The joint probability density of \((S_{k-1}, S_k)\) is

\[
\frac{1}{\sqrt{2\pi V_{k-1}} \sqrt{2\pi V_k}} \exp \left[ -\frac{(s_{k-1} - M_{k-1})^2}{2V_{k-1}} \right] \frac{1}{\sqrt{2\pi V_k}} \exp \left[ -\frac{(s_k - s_{k-1})^2}{2\sigma_k^2} \right],
\]

for any \( k > 1 \).

**Lemma 2.** Let \( X_1 \sim N(\mu_1, \sigma_1^2) \), \( X_k \sim N(\mu_k, \sigma_k^2) \), \( k = 2, 3, \ldots \), be independent identically distributed variables, and \( W = (w_1, w_2, w_3, \ldots) \), be nonnegative values. Let \( S_k = \sum_{i=1}^{k} X_k \), and \( W_K = \sum_{k=1}^{K} w_k \). For any nonnegative \( w \) and \( k = 2, 3, \ldots, S_k \) satisfies the Markov condition, namely,

\[
P(S_k > W_k | S_1 > W_1, \ldots, S_k > W_{k-1}) = P(S_k > W_k | S_{k-1} > W_{k-1}).
\]

\( \{S_k\} \) is an independent incremental process, providing that \( S_{k-1} \), \( S_k \) is just dependent on \( X_k \), which is independent of any \( X_i \) for \( i = 1, 2, 3, \ldots, k-1 \). Hence, \( S_k \) is independent of \( S_i \) for \( i = 1, 2, 3, \ldots, k-1 \) given \( S_{k-1} \).

**Theorem 1.** Let \( X_1 \sim N(\mu_1, \sigma_1^2) \), \( X_k \sim N(\mu_k, \sigma_k^2) \), \( k = 2, 3, \ldots \), be independent identically distributed variables, \( W = (w_1, w_2, w_3, \ldots) \), be nonnegative values, \( S_k = \sum_{i=1}^{k} X_k \), \( M_K = \sum_{k=1}^{K} \mu_k \), \( V_K = \sum_{k=1}^{K} \sigma_k^2 \), and \( W_K = \sum_{k=1}^{K} w_k \). Let \( P_k = P(T > k) = P(S_k > W_k) \) for any \( k = 1, 2, 3, \ldots \), and \( P_{k|k-1} = P(S_k > W_k | S_{k-1} > W_{k-1}) \). Then we have

(a) \( P_1 = P(T > 1) = P(X_1 > W_1) \).

(b) \( P_k = P_1 \prod_{i=2}^{k} P_{k|i-1} \).

(c) For any \( k = 2, 3, \ldots \),

\[
P_{k|k-1} = \frac{1 - \Phi(A_{k|1}) - \int_{A_{k|1}}^{\infty} p(z) \Phi \left( \frac{w_k - M_k - \sqrt{V_k}}{\sigma_k} \right) dz}{1 - \Phi(A_{k|1})},
\]

where \( A_{k|1} = \frac{w_k - M_k - \sqrt{V_k}}{\sigma_k} \), \( p() \) and \( \Phi() \) are the probability density function and cumulative distribution function of a normal distribution, respectively.

**Proof**

(a) The first result \( P_1 = P(T > 1) = P(X_1 > W_1) \) can be derived from its definition directly.

(b) From the definition of \( P_k \),

\[
P_k = P(T > k)
\]

\[= P(S_1 > W_1, \ldots, S_k > W_k) \]

\[= P(S_1 > W_1, \ldots, S_{k-1} > W_{k-1}) \times P(S_k > W_k | S_1 > W_1, \ldots, S_{k-1} > W_{k-1}).
\]

According to Lemma 2, the term of conditional probability can be simplified to \( P(S_k > W_k | S_{k-1} > W_{k-1}) \). It follows that \( P_k \) can be factorized to

\[
P_k = P_{k-1} P(S_k > W_k | S_{k-1} > W_{k-1}) = P_{k-1} P_{k|k-1}.
\]

Following the same factorizations from \( P_{k-1} \) to \( P_2 \), we obtain

\[
P_k = P_1 \prod_{i=2}^{k} P_{i|i-1}.
\]

(c) Let \( Z_1 = \frac{s_{k-1} - M_{k-1}}{\sqrt{V_{k-1}}} \), and \( Z_2 = \frac{s_k - s_{k-1} - M_k}{\sqrt{V_k}} \). From the definitions of the \( S_k \), it is concluded that \( Z_1 \) and \( Z_2 \) are independent variables and follow the standard normal distribution. Solving \((S_{k-1}, S_k)\) from the above two equations leads to \( S_{k-1} = M_{k-1} + \sqrt{V_{k-1}} Z_1 \) and \( S_k = M_k + \sqrt{V_k} Z_2 + \sigma_k Z_2 \). In order to calculate \( P_{k|k-1} \), we need to work out two probabilities, which are \( P(S_k > W_k, S_{k-1} > W_{k-1}) \) and \( P(S_k > W_k) \). Lemma 1 shows the distribution of \( S_k \) and the joint distribution of \((S_{k-1}, S_k)\). A two-dimensional integral is needed to get the probability of \( P(S_k > W_k, S_{k-1} > W_{k-1}) \). By replacing \((S_{k-1}, S_k)\) with \( Z_1, Z_2 \) in the integral transfer, we get

\[
P_{k|k-1} = \frac{1 - \Phi(A_{k|1}) - \int_{A_{k|1}}^{\infty} p(z) \Phi \left( \frac{w_k - M_k - \sqrt{V_k}}{\sigma_k} \right) dz}{1 - \Phi(A_{k|1})},
\]

where \( A_{k|1} = \frac{w_k - M_k - \sqrt{V_k}}{\sigma_k} \), \( p() \) and \( \Phi() \) are the probability density function and cumulative distribution function of normal distribution, respectively. □

**Corollary 1.** \( P_k \) and \( P_{k-1} \) have the following relation

\[
P_k = P_{k-1} P_{k|k-1}
\]

for \( k = 2, 3, \ldots \).

**Theorem 1** presents a numerical method to calculate the probabilities of events with stopping time. As we have shown previously, in the pricing model, the expected revenue of renting the search engine is a function of probabilities of stopping time. Hence, we have solved the crux by working out the probability \( P_k \). Under the framework of
the pricing model, different pricing strategies are imposed with the constraints with different parameters. For example, the parameter constraints

\[ \mu_1 = D\mu > 0, \quad \sigma_1^2 = D\sigma^2 > 0, \]

\[ \mu_2 = \mu_3 = \cdots = \mu_H = \mu > 0, \quad \sigma_2^2 = \sigma_3^2 = \cdots = \sigma_H^2 = \sigma^2 > 0, \]

\[ w_1 = w_2 = \cdots = w_H = aw > 0, \]

(34) (35) (36) can be used to model the strategy of offering a coupon with the constraints with different parameters. For example, the parameter constraints

\[ \mu_1 = D\mu > 0, \quad \sigma_1^2 = D\sigma^2 > 0, \]

\[ \mu_2 = \mu_3 = \cdots = \mu_H = \mu > 0, \quad \sigma_2^2 = \sigma_3^2 = \cdots = \sigma_H^2 = \sigma^2 > 0, \]

(37) where \( p(\cdot) \) and \( \Phi(\cdot) \) are the probability density function and cumulative distribution function of normal distribution, respectively.

From the conditions, we have \( M_k = (D + k - 1)\mu \), \( V_k = (D + k - 1)\sigma^2 \), and \( W_k = a(D + k - 1)w \). By setting them into \textit{Theorem 1}, we obtain the following expected revenue of renting searching engine.

\textbf{Corollary 3.} The expected revenue is a function of the monthly rental, given by

\[ Q(w) = \sum_{k=1}^{D} d^k w + w \sum_{k=2}^{H-D+1} d^{k-D-1} P_k \prod_{i=2}^{k} P_{k|k-1} \]

(45) in the model for strategy of initial lock-in period.

From the definition, the expected revenue is a function of monthly rental. When we set the rental to a value, the expected revenue can be easily numerated since the probabilities about stopping time in the expected revenue only contain two-dimensional integral from the above corollary. According to \textit{Corollaries 1–3}, we further list an algorithm to calculate the expected revenue given a certain monthly rental.

\textbf{Algorithm 1}

\begin{enumerate}
  \item Let \( Q_1 = \sum_{k=1}^{D} d^k w \), \( P_1 = 1 - \Phi\left[\frac{\sqrt{D}(aw - \mu)}{\sigma}\right] \).
  \item From \( k = 2 \) to \( H - D + 1 \), calculate \( P_k = P_{k-1} P_{k|k-1} \) and \( Q_k = Q_{k-1} + w d^{D+1-k} P_k \).
  \item Let \( Q(w) = Q_{H-D+1} \).
\end{enumerate}

If we set the parameters \( D, R, \) and \( H \), searching the best monthly rental is just a one-dimensional optimization problem. Any general optimization method, say, Newton-Raphson algorithm, can be used to search the best monthly rental.

\subsection*{4.2. Best rental for strategy of offering a coupon in the first month}

Similar to the case in the model for strategy of initial lock-in period, through setting some specific parameters in \textit{Theorem 1}, we obtain the \textit{Corollary 4} for strategy of offering a coupon in the first month. It presents us a numerical method to calculate the expected revenue in the model for strategy of waiving certain amount of fee in the first month.

\textbf{Corollary 4.} Let \( X_k \sim N(\mu, \sigma^2), k = 1, 2, 3, \ldots, \) be independent identically distributed variables, and \( S_k = \sum_{i=1}^{k} X_k \). Let \( w \) and \( v \) be nonnegative and \( w > v, a > 1 \) and \( P_k = P(T > k) \), and for any \( k = 2, 3, \ldots, \)

\[ P_{k|k-1} = P[S_k > a(kw-v)]S_{k-1} > a[(k-1)w-v]], \]

(46) and

\[ A_{k} = \frac{(k-1)(aw-\mu) - av}{\sqrt{k-1} \sigma}. \]

(47)
We have
\[ (a) \quad P_1 = P(T > 1) = P(X_1 > w - v) = 1 - \Phi \left( \frac{aw - av - \mu}{\sigma} \right). \]
\[ (b) \quad P_k = P_1 \prod_{i=2}^{k} P_{\ell_{k-1}}. \]
\[ (c) \quad \text{For any } k = 2, 3, \ldots, \]
\[ P_{\ell_{k-1}} = \frac{1 - \Phi(A_{1i}) - \int_{A_{1i}}^{\infty} \phi(z) \phi \left( \frac{aw - av - kw - \gamma}{\sigma} \right) dz_1}{1 - \Phi(A_{1i})}. \]

From the conditions, we have \( M_k = k\mu, \) \( V_k = k \sigma^2, \) and \( W_k = a (kw - v). \) Set them into Theorem 1, we obtain the expected revenue of renting searching engine.

**Corollary 5.** The expected revenue in the model for strategy of offering a coupon in the first month is a function of the monthly rental as

\[ Q'(w, v) = wE \left[ \sum_{i=1}^{H-1} d^i P_i \prod_{i=2}^{k} P_{\ell_{k-1}} \right] - v. \]  

Here, we set \( \prod_{i=2}^{k} P_{\ell_{k-1}} \) to 1 if \( k = 1. \)

The expected revenue in this model is a function of monthly rental and coupon. When we set the rental and coupon to some certain values, the expected revenue can be easily numerated since the probabilities about stopping in the expected revenue only contain two-dimensional integral from the above corollary. According to Corollaries 1, 4 and 5, we list an algorithm to calculate the expected revenue given a certain monthly rental \( w \) and the coupon \( v. \)

**Algorithm 2**

Step 1 Let \( P_1 = 1 - \Phi \left( \frac{aw - av - \mu}{\sigma} \right), Q'_1 = wdP_1 - v. \)

Step 2 From \( k = 1 \) to \( H - 1, \) calculate \( P_k = P_{k-1}P_{\ell_{k-1}} \) and \( Q'_k = Q'_{k-1} + wdP_k. \)

Step 3 Set \( Q'(w, v) = Q_{H-1}. \)

If we set the parameters \( R, \) and \( H, \) searching the best monthly rental and best coupon is just a two-dimensional optimization problem with constrain condition \( 0 \leq v \leq w. \) Any general conditional optimization method can be used to search the best monthly rental and the best coupon.

**5. Illustrative examples and discussions**

Assume that the interest rate is \( p. \) Then the monthly discounter factor is \( d = \frac{1}{1 + \frac{p}{12}} = \frac{12}{12 + p}. \) If \( p = 5\%, \) \( d = 0.9959. \) Let \( \mu = 20,000, \sigma = 10,000, \theta = 0.5, H = 36 \) months.

In this section, we investigate the experiments for the effects of different strategies by allowing different lengths of initial lock-in periods and offering the waiver of different amounts of fee for initial usage in the first month.

**5.1. Best rental for strategy of initial lock-in period**

For the initial lock-in pricing strategy, we can get several properties about the model. First, in the beginning, there is
no accumulative saving of renting the search engine. Second, the saving and cost are both at a lower level. Third, the variation of saving of the lessee has more powerful effect in the first month than that in the latter months. Those properties induce the fact that in the first month, the lessor is more apt to lose the lessees. As discussed in Section 2, the initial lock-in pricing strategy can decrease the probability that lessees quit. In Fig. 2, different lengths of initial lock-in periods, two months, three months, six months, nine months and one year are considered to discuss the relationship of rental and revenue. Five rental–revenue curves are shown in Fig. 2. It is found that all of the five pricing strategies reach the optimal monthly rentals in the interval from 5000 to 10,000. It is concluded that the longer the initial lock-in period is, the bigger the optimal monthly rental will be.

Table 1 shows the optimal monthly rental \( w^* \) within the interval from 0 to \( \frac{a}{2} = 10,000 \) and the corresponding rental revenue \( Q(w^*) \) with different initial lock-in periods and different interested periods.

5.2. Best rental for strategy of offering a coupon in the first month

The strategy of offering a certain amount of coupon in the first month decreases the probability of lessee quitting in the beginning. Fig. 3 illustrates five rental–revenue curves with five different coupons in the first month, no coupon, a quarter of rental, half of rental, three quarters of rental, and the rental. In the first scenario, no coupon is offered and in the last scenario, the lessee can utilize the search engine free. Fig. 3 shows the five pricing schemes with the optimal monthly rentals, which are in an interval from 6000 to 10,000.

Fig. 4 illustrates the relationship of the coupons in the first month and revenues in the different monthly rental levels. We set monthly rental as 1000, 3000, 5000, 7000, 10,000, respectively and assume the coupon in the first month is less than the monthly rental. We can find that when monthly rental is at a lower or higher level, say

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<tr>
<td>( Q(w^*) )</td>
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Fig. 5. Curves of optimal monthly rentals and the best revenues. Units in \( x \) and \( y \) axes are US dollars.
1000 or 10,000, the coupon in the first month has little effect on the revenue. When monthly rental is 1000, the coupon with a larger value will decrease the revenue. When monthly rental is at a proper level, say 7000, a number near the optimal monthly rental, the coupon in the first month can improve the revenue significantly.

Table 2 shows the optimal coupons in the first month, best monthly rental \( w^* \) in the interested period, and the corresponding \( Q(w^*) \). The optimal first-month-coupon is equal to the best monthly rental, namely, the lessees utilize the search engine free in the first month.

5.3. Compare two strategies

We have obtained the optimal monthly rental and corresponding revenue for each interested period. Fig. 5 shows the optimal monthly rentals and revenues of the two pricing strategies. One is the initial lock-in period strategy with \( D = 3 \), the other is to offer a coupon in the first month. In Fig. 5, there are three axes, one horizontal axis and two vertical axes. The horizontal coordinate denotes the interested period, the left vertical axis symbolizes for the best revenue, and the right vertical axis stands for the optimal monthly rental. There are four curves in the figure, two for initial lock-in period strategy with \( D = 3 \) (solid lines) and two for coupon-offering strategy (dashed lines). We can find that the optimal monthly rental is a decreasing function of the interested period. Compared to coupon-offering strategy, the initial lock-in period strategy leads to a lower optimal monthly rental and lower revenue.

The average renting periods for different pricing strategies and interested periods, \( \frac{Q(w^*)}{w^*} \), are shown in Table 3. From Table 3, we can find that the coupon strategy has the shortest average renting time.

6. Conclusion

The pricing problem for local search engines is formulated, analyzed and experimented. The model is formulated by analyzing the independent incremental process. The optimal rental is found experimentally based on the derivative of the expected present value of the purchasing price.

Pricing a search engine involves in many complex factors and thus difficult to analyze. In this paper, the model is only formulated based on analyzing the search engine users’ saving and search engine providers’ profit maximization. It should be admitted that this fundamental assumption is of many limitations. For example, the reservation value for search engine users may be hard to obtain, and different search engine users may have different reservation values. In addition, it may not be so easy to obtain the total costs of search engine users in practice. However, the problem formulating and solving procedure highlights one of exploring ways of pricing search engines in the web era.

With the development of the eCommerce, an increasing number of websites incorporate the search engines. In response to that, the model in this paper provides an effective tool for pricing them.

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